The essence of integration

I. The core geometric essence of integration: finding the area under a function curve.

For a continuous function $y=f(x)$ defined on the interval $[a, b]$ (without loss of generality, assume $f(x) \geq 0$), the core geometric meaning of the integral is to find the area of the curvilinear trapezoid enclosed by the curve $y=f(x)$ and the lines $x=a$, $x=b$, and the x- axis.

The boundary of this curvilinear trapezoid is composed of curve segments and straight line segments. Because the function curve has "concave and convex changes", it cannot be directly calculated using the area formulas of rectangles and trapezoids in elementary geometry. The core solution is to approximate the complex irregular shape with a simple regular shape and eliminate the error through "infinite subdivision" .

II. Decomposition of a Curved Trapezoid: Combining an Infinite Number of Narrow Rectangles

The key to transforming the area of a curvilinear trapezoid into a calculable quantity lies in breaking down the overall area into the sum of the areas of an infinite number of infinitely small narrow rectangles . The specific decomposition logic is as follows:

1. Subdividing the interval : dividing the domain of the function into intervals $[a, ..., ...]$ $b]$ The partition can be arbitrarily divided into n small intervals, and the width between each small interval is denoted as Δx . When the number of partitions $n \to \infty$, the width Δx between small intervals approaches infinitesimal, denoted as dx (that is, the

differential of the independent variable x , with the base dx).

2. straight lines for curves : For each narrow interval with a width of dx , take the x- coordinate of any point within the interval , and use the function value $f(x)$ of that point as the height of the narrow rectangle . At this time, the concavity and convexity of the curve within the narrow interval can be ignored. The straight edge of the rectangle is used to replace the curved edge of the function, thus achieving "substituting straight lines for curves".
3. Area of a single block : The area of each narrow rectangle is base × height , i.e., $dS = f(x) \cdot dx$ (dS is an infinitesimal area element).
4. Cumulative summation : The total area of the curvilinear trapezoid is the sum of the area elements of all narrow

rectangles in the interval $[a, b]$. The cumulative sum over $[b]$ – when the number of narrow rectangles is infinite, this cumulative sum is the essence of integration.

III. The Origin of the Integral Symbol: From Finite Summation to Infinite Summation

1. In elementary mathematics, we use $\sum$ (sigma) represents the summation of a finite number of terms, such as $\sum^{5}$ k=1 k=1+2+3+4+5 . However, this notation is only applicable to "discrete, finite" summation scenarios and cannot characterize "continuous, infinite" accumulations, thus having limitations.
2. To represent the infinite summation of continuous quantities , calculus introduced the integral symbol " $\int$ " – which evolved from sigma $\sum$ and is essentially a "stretched

sigma", specifically used to represent the "cumulative sum of an infinite number of infinitesimals".

3. The area of a curvilinear trapezoid can be expressed as an integral: $S = \int_a^b f(x)dx$

Where: $\int$ is the integral symbol (representing infinite summation), a and b are the upper and lower limits of integration (representing the range of summation), $f(x)$ is the integrand (representing the height of the narrow rectangle), and dx is the integration variable (representing the base of the narrow rectangle, i.e., the infinitesimal increment of the independent variable).

The essential distinction between integrals and geometric areas and physical quantities

I. Core Conclusion: Integrals can characterize area, but integrals ≠ geometric area.

The geometric meaning of integrals can be intuitively represented by the area under the function curve . However, as a signed cumulative sum , the result of integrals is fundamentally different from that of geometric areas: geometric areas are always positive scalars and only involve "accumulation"; while integrals are signed operations and can "cancel out" due to the sign of the function value . Therefore, the result of integrals cannot be directly derived into geometric areas.

The contradiction between integrals and geometric area: the canceling effect caused by the positive and negative signs of function values.

For the interval [a, ..., ...] A continuous function y=f(x) on [c]

exists if there exists b $\in$ (a, c). c) , satisfying:

- x $\in$ [a, When b] , f(x) < 0 (the curve is below the x- axis);
- x $\in$ [b, When c] , f(x)>0 (the curve is above the x- axis).

At this point, the function is in the interval. The integral over

[a, c] is: $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$

below the x- axis $\int_a^b f(x)dx$ is negative , and the integral

above $\int_b^c f(x)dx$ the x- axis is negative.

If the result is positive , the two will cancel each other out,

and the result of the integration is the net cumulative value ,

rather than the total geometric area enclosed by the curve

and the x- axis .

Key difference : Geometric area is directionless and unsigned,

requiring only full summation; integration is a signed

cumulative operation , where the sign of the function value corresponds to the sign of the integral, resulting in a canceling effect. Therefore, the integral result cannot be directly equated to the geometric area.

II. Physical Examples to Support the Evidence: The Correspondence Between Integral Values and Displacement/Distance in a vt Graph

The "signed accumulation and cancellation of positive and negative signs" property of integrals is particularly evident in kinematics. Taking vertical upward projectile motion (a soccer ball rising and falling) as an example, and combining it with the v−t (velocity - time) graph from the IB physics core, we can clearly understand the physical meaning of the integral and its relationship with the geometric area:

1. Sports Background

When a soccer ball is kicked vertically upwards, it first undergoes uniformly decelerated linear motion upwards (its speed gradually decreases until it reaches zero at its highest point), and then undergoes uniformly accelerated linear motion downwards (its speed increases in the opposite direction and eventually falls back to the ground). The change in velocity over time during the entire process can be described by a v - t graph.

2. The integral meaning of the vt graph (core)

In the v−t image:

- The velocity v(t) is the vertical axis, and the time t is the horizontal axis. The upward direction is defined as the positive direction of velocity , and the downward direction is defined as the negative direction of velocity.

- The signed area enclosed by the image and the time axis is the integral $\int_{t_1}^{t_2} v(t)dt$, and the result corresponds to the physical quantity displacement .
- The total geometric area enclosed by the image and the time axis (the sum of the absolute values of all parts) corresponds to the physical quantity distance .

3. The cancellation effect of integration and the essential distinction between physical quantities

- Displacement : A vector quantity, with direction and sign, reflecting the net change in the position of an object .

  The soccer ball starts from the ground, rises into the air, and falls back to the ground. In the v - t graph of the whole process, the integral during the rising phase (the velocity is positive) is positive, and the integral during the falling phase (the velocity is negative) is negative. Since the two are equal in magnitude and opposite in direction , they completely cancel each other out , and the final integral result is 0, which corresponds to the soccer ball's displacement being 0 (no net change in position).

- Distance : a scalar quantity, without direction, always positive, reflecting the actual path length of an object's motion .

  The actual trajectory of the soccer ball is "distance of ascent + distance of descent", which corresponds to the sum of the geometric areas of the ascent and descent phases in the v−t graph (unsigned cancellation, full summation). This result is completely different from the net cumulative value of the integral.

The underlying essence of derivatives: the geometric meaning of dy /dx and the independence of differentials.

I. The origin of the derivative dy /dx : from the incremental ratio to the slope of the tangent line

The definition of the derivative originates from the ratio of the increments of a function , and its core geometric meaning is "the slope of the tangent line after infinite approximation," which can be specifically broken down as follows:

1. For a continuous function $y=f(x)$, take two points $P(x, y)$ and $Q(x+\Delta x , y+\Delta y)$ on the curve , where Δx is a finite increment of x and Δy is a corresponding finite increment of y ;

2. a secant line of the curve , and the slope of the secant line is the increment ratio . $k_{割线} = \frac{\Delta y}{\Delta x}$

3. As point Q approaches point P infinitely along the curve , the finite increments Δx→0 and Δy→0 become infinitesimal differentials , denoted as Δx→dx and Δy→dy .

4. The secant line, as it infinitely approximates the two points, becomes the tangent line to the curve at point P. The slope of the tangent line is the derivative, i.e.:

   $$f'(x) = \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

   Key conclusion : The essence of dy /dx is the ratio of infinitesimal differentials , not simply a derivative notation; dy is the infinitesimal differential of y , and dx is the infinitesimal differential of x . The ratio of the two describes the instantaneous rate of change of the function at a certain point, which is geometrically represented by the slope of the tangent line.

II . Core Question: Can dy / dx be reversed to dx / dy ?

1. Exclude "coordinate axis rotation"

If we invert dy / dx to dx / dy , it 's not simply a matter of placing the x - axis vertically and the y- axis horizontally (a superficial operation of differentiating x with respect to y). What we want to explore is: in the original coordinate system, does dx / dy , as the reciprocal of dy / dx , have a direct geometric meaning?

2. Geometric meaning of dx / dy : The slope of the normal to the curve at that point.

The answer is clear: dx / dy is the function curve at point P(x, The slope of the normal at point y , and the normal and the tangent have a core geometric relationship , that is, they are perpendicular to each other .

- The slope of the tangent line is k_tangent = dy / dx, and the slope of the normal line is k_normal = dx / dy .
- From the slope relationship of perpendicular lines (the product of slopes is -1 when not perpendicular to the coordinate axes), it can be verified that: ktangent * knormal = dy / dx * dx / dy = 1

  Core Relationship : The reciprocal relationship between dy / dx and dxdy essentially corresponds to the perpendicular relationship between the tangent and the normal of the curve at a certain point . This is the most direct geometric meaning of dy / dx after inverting.

III. Fundamental Root Cause: Why can dy and dx be reversed and separated?

that dy/dx can be arbitrarily inverted to dx/dy is primarily because the differentials dy and dx are independent infinitesimals , not bound global notations. The root of this independence can be traced back to the infinitely narrow rectangular model we used in our previous analysis of integrals :

1. In the integral, we decompose the curvilinear trapezoid into infinitely narrow small rectangles , with width dx (an infinitesimal differential of x), height $f(x)$, and area $dS = f(x) \cdot dx$.
2. For this infinitely narrow rectangle, dx is an independent infinitesimal (which can be understood as an "infinitely small number"). Similarly, from $dy = f'(x) \cdot dx$, we know that dy is another independent infinitesimal determined by dx and the derivative .

3. As independent infinitesimals , dy and dx have the same operational properties as ordinary numbers . The positions of the numerator and denominator can be freely interchanged in a fraction, and they can also be separated (e.g., transformed into $dy = f'(x) \cdot dx$).

Key conclusions :

dy and dx can be "inverted and separable" is not an artificial rule of calculus, but rather stems from their independence . They are two independent infinitesimals, essentially "numbers that can participate in operations." Therefore, they can be combined to form dy /dx to represent the slope of the tangent line, or inverted to dx / dy to represent the slope of the normal line. They can also be separated into $dy = f'(x) \cdot dx$ to perform separate differential operations.

Clarifying the Essence of the Second Derivative and the Mathematical Definition of Perpendicularity

The essence of the first and second derivatives: symbolic definition and geometric meaning

1. The core meaning of the second derivative: the rate of change of the derivative.

The first derivative $f'(x) = dy/dx$ describes the instantaneous rate of change (slope of the tangent line) of the function $y=f(x)$. The second derivative, as the derivative of the first derivative , measures the rate of change of the slope of the tangent line , that is, how fast the function changes.

- If $f''(x)>0$, the first derivative dy/dx is monotonically increasing, the slope of the tangent line increases with x , and the function curve is concave upward .
- If $f''(x)<0$, the first derivative dy/dx is monotonically decreasing, the slope of the tangent line decreases as x increases, and the function curve is concave downwards .

In layman's terms, the second derivative reflects whether the trend of a function is " getting better " (the slope gets bigger and the curve bends upward) or " getting worse " (the slope gets smaller and the curve bends downward), which is the core reason why it describes the concavity and convexity of a curve.

2. Symbolic definition of the second derivative: d^2 The essence of y/dx^2 (not squared)

The rigorous expression of the second derivative is derived from the differentiation of the first derivative, and its derivation in sign form is directly related to the independence of differentials:

1. The second derivative is the derivative of the first derivative dy /dx with respect to x , and is strictly written as: $f''(x) = \frac{d}{dx}\left(\frac{dy}{dx}\right)$
2. The official simplified notation is d ^{2} In the notation y/dx ^{2} , the "2"s above and below do not have a square meaning .
   - molecule d2 y : indicates that y is differentiated twice . The first d is the sign of the first derivative, and the second d corresponds to the differential dy of the original function .

- The denominator dx^2 represents the derivative of x twice , abbreviated as $dx \cdot dx$, which is the abbreviated form of $(dx)2$, and is not the square of x .

3. The core origin of the notation: the independence of the differential dx

If dx is not considered as an independent infinitesimal , it is very easy to mistake d^2 for an infinitesimal · The y/dx^2 is mistakenly read as " $d^2 y$ divided by $(dx)^2$ " , which is a typical terminology error in calculus.

It is precisely because dx is an independent, computationally comprehensible infinitesimal that we can decompose the process of continuous differentiation into the operational form of differentials, thus clearly defining $d^2 \cdot y/dx^2$ is an abbreviation for " differentiating y twice and taking the derivative twice with respect to x ", not a squaring operation.

II. Introduction through a physical example: The projectile motion of a badminton shuttlecock at high altitude

1. Motion model (ideal scenario, indoors with no wind)

A badminton shuttlecock is thrown obliquely upwards at an initial velocity u at an angle of 60° to the horizontal . The motion of the shuttlecock is projectile motion , and its trajectory is a combination of horizontal and vertical components:

- Horizontal direction: Uniform linear motion (no external horizontal force, velocity remains constant);
- Vertical direction: Vertical upward projectile motion (only subject to gravity, uniformly accelerated linear motion).

- Variable association: The horizontal displacement x and vertical displacement y of the badminton shuttlecock are both functions of time t, i.e., $x = x(t)$ and $y = y(t)$.

2. Problem Statement

Given that the tangents to the trajectory of a badminton shuttlecock at the 5th second ($t= 5$) and the 15th second ($t= 15$) are perpendicular, find the magnitude of the initial velocity u.

III. Two Mathematical Definitions of Perpendicularity: Clarification and Applicable Scenarios

Definition 1: Definition of perpendicularity of vectors (general type, no scene restrictions)

Core conclusion : If two non-zero vectors a = (x1,y1) , b=(x2, y2) are perpendicular to each other, then their dot product is equal to 0 , that is: $\vec{a} \cdot \vec{b} = x_1x_2 + y_1y_2 = 0$

- In essence: The definition of perpendicularity of vectors is a coordinate-independent general definition that only depends on the components of the vector and does not rely on lines, equations, or fixed coordinate systems.
- For example, the dot product of unit vector i = (1,0) (horizontally to the right) and vector j = (0,1) (vertically upward) is 1×0+0×1=0 , so they are perpendicular.

Definition 2: Definition of perpendicularity of a line (restricted, applicable only to lines)

Key conclusion : If two lines not perpendicular to the coordinate axes are perpendicular to each other, then the product of their slopes is equal to -1 . That is, if the slope of line l_1 is k_1 and the slope of line l_2 is k_2 , then: $k_1 \cdot k_2 = -1$

- Prerequisites for application:
    1. It must be a straight line ;
    2. A fixed coordinate system is required to determine the slope of a straight line by its equation; without a coordinate system, there is no slope to speak of.

The core difference between the two definitions: the relationship between the coordinate system and the variables. The definition of a perpendicular line (slope product = -1) depends on a fixed coordinate system , while the coordinate system we establish for the oblique projectile motion of a badminton shuttlecock has clear characteristics:

- With the point where the shuttlecock is thrown as the origin , the positive x- axis is horizontally to the right and the positive y- axis is vertically upward .

- In this coordinate system, the horizontal coordinate x represents the horizontal displacement, and the vertical coordinate y represents the vertical displacement. Both are functions of time t , but the coordinate system itself is not explicitly related to time t —the coordinate system is fixed, and time t is the independent variable hidden behind x and y .

IV . Step 1: Decompose the formulas for velocity and displacement in the horizontal/vertical directions.

The core of projectile motion is that the horizontal and vertical components of motion are independent . By combining the decomposition of the initial velocity, we can derive the expressions for x(t) and y(t) , as well as the component velocities dx/dt and dy / dt .

1. Orthogonal decomposition of initial velocity (60° included angle)

- Horizontal velocity component (x direction): U x = ucos60 ° = 1/2 u (horizontal uniform velocity, unchanged throughout the entire process)
- Vertical velocity component (y- direction): Uy = usin60 ° = √ 3 /2 u (vertical uniform acceleration, affected by gravity)

2. Functional relationship between displacement and time (parametric equation, independent variable is t)

- Horizontal displacement (uniform linear motion): x(t) = 1/2 u · t
- Vertical displacement (vertical upward projectile motion, displacement formula:)

$s = v_0 t - \frac{1}{2} g t^2$)： $\boldsymbol{y(t) = \frac{\sqrt{3}}{2} u \cdot t - 5t^2}$ （因$\frac{1}{2}g = 5$）

3. The derivative of the displacement with respect to time (partial velocity, core).

The physical meaning of the derivative: The derivative of displacement with respect to time is the instantaneous velocity, obtained by directly differentiating the parametric equations.

- Horizontal direction: $dx/dt = 1/2\ u$ (horizontal velocity is constant, derivative is constant)
- Vertical direction: $dy/dt = \sqrt{3}/2\ u - 10t$ (vertical uniform deceleration, derivative is a linear function, 10 is the gravitational acceleration g)

Step 2: Core Derivation

1. Derivation basis: Independence of differentials

x and y have no direct functional relationship, both being related to time t . However, dt is an independent infinitesimal differential (which can be used as a "movable denominator" in arithmetic operations). Therefore, dy /dx can be

transformed as follows: $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$

Essentially, the core principle of parametric function differentiation is to transform the differentiation with respect to x into the differentiation with respect to the common parameter t , and then use the independence of differentials to perform ratio operations.

2. Substitute the component velocities to find the general formula for dy /dx.

Substituting dx /dt = 1/2 u and dy /dt = √ 3 /2 u−10t into the

formula, we get $\frac{dy}{dx} = \frac{\frac{\sqrt{3}}{2}u-10t}{\frac{1}{2}u}$

Step 3: Substitute the perpendicularity condition and solve for the initial velocity u.

1. Mathematical conditions for perpendicularity

t=5 and t=15 , therefore the product of their slopes = -1 .

$$\left.\frac{dy}{dx}\right|_{t=5} \cdot \left.\frac{dy}{dx}\right|_{t=15} = -1$$

2. Find the slopes

- 当$t = 5$时：$k_5 = \frac{\frac{\sqrt{3}}{2}u - 10\times 5}{\frac{1}{2}u} = \frac{\frac{\sqrt{3}}{2}u - 50}{\frac{1}{2}u}$
- 当$t = 15$时：$k_{15} = \frac{\frac{\sqrt{3}}{2}u - 10\times 15}{\frac{1}{2}u} = \frac{\frac{\sqrt{3}}{2}u - 150}{\frac{1}{2}u}$

at t=5 and t=15 respectively.

3. Substitute the perpendicularity condition to simplify the calculation.

Substituting k5 and k15 into the condition that the product is -1, since the denominators are both 1/2 u , we can first cancel the denominators (u≠0 , the initial velocity is not 0), and simplify to obtain the core equation:

$$\left(\frac{\sqrt{3}}{2}u - 50\right) \cdot \left(\frac{\sqrt{3}}{2}u - 150\right) = -\left(\frac{1}{2}u\right)^2$$

Trigonometric derivatives and the method of integrating differentials

The differential form of the derivative is denoted as d() , where the function following d is the object being differentiated.

The essence of integration is to find an antiderivative that, when differentiated, equals the integrand , which is essentially stuffing the expression after d .

E g: Solve for $\int \sin^3 x \cdot \cos x \, dx$

- Step 1: Observe the expression and identify the part where d can be hidden.

Integrand: $\sin^3 x \cdot \cos x \, dx$

Only $\cos x \, dx$ perfectly matches $d(\sin x)$, which is the only breakthrough in solving the problem.

- Step 2: Substitution Simplification

Let $\sin x = \triangle$

The original expression is reduced to: $\int \triangle^3 \cdot d\triangle$, which

becomes the simplest polynomial integral .

- Step 3: Inverse operation of polynomial integration

(inverse differentiation)

We know the differentiation rule: $(\triangle^4)' = 4\triangle^3$

Integrating is the inverse process of differentiation, so we

need to divide by a correction factor of 4 :

$\int \triangle^3 \, d\triangle = (1/4)\triangle^4$

- Step 4: Retrospective Restoration

Substituting $\Delta = \sin x$ back into the equation, the final result

is: $(1/4) \sin^4 x + C$

* chain rule

dx and dy. The operation of "hiding the function after d" is a direct manifestation of the freedom of the four arithmetic operations of differential.

From Image Fitting to Taylor Expansion and the Essence of Integrals

- Taylor's formula : Any function can be written as

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

- Taylor's significance

Break down the messy functions into a bunch of polynomials and add them together , and use the tools we are most familiar with to handle the most annoying functions.

- McLaughlin

McLaughlin's formula is the Taylor formula with the expansion point $a=0$, which is useful but has limitations .

- counterexample

For example , $\ln(x)$ and $\ln(0)$ do not exist, so we must use generalized Taylor.

"Guess" $\ln(1-x)$ using an infinite polynomial.

1. Infinite geometric sequence sum

   We know the simplest infinite polynomial:

   $1 + x + x^2 + x^3 + \dots = 1 - x^1$

   (Formula for sum of infinite terms: first term / (1 - common ratio))

2. Grasp the derivative relationship

   We know that the derivative of $\ln(x)$ is $1/x$, so the integral of $1/(1-x)$ must be related to $\ln$.

3. C hain rule error correction

   The direct product 1/(1-x) :

   My instinct is that it's ln(1-x) , but I verified it using the chain rule:

   $[\ln(1-x)]' = 1-x-1$

   There's an extra negative sign (because the derivative of 1-x with respect to x is -1, which is not clean).

   Therefore, the corrected result is: the integral result is -ln(1-x) .

4. Polynomial fitting confirmed

   Left side: Integrating term by term of an infinite polynomial and then adding them together;

   Right side: Directly integrate -ln(1-x)

   The conjecture is true.

Why can points be split freely?

The integral symbol $\int$ essentially represents infinite summation.

Addition satisfies the commutative and associative laws, so summation can be achieved by freely splitting and combining the elements.

The verification of L'Hôpital's rule and Taylor's method

Solving limit operations : $\lim_{x \to 0} \frac{\sin x}{x}$

1. L'Hôpital's rule for type 0/0 : Fundamental theorem

Suppose that functions f(x) and g(x) satisfy the following conditions:

1. $\lim_{x \to a} f(x) = 0, \ \lim_{x \to a} g(x) = 0;$

2. In some deleted neighborhood of point a , both f′(x) and g′(x) exist, and g′(x) ≠0 ;
3. $\lim_{x\to a}\frac{f'(x)}{g'(x)}$存在（或为无穷大）；

则极限$\lim_{x\to a}\frac{f(x)}{g(x)}$与$\lim_{x\to a}\frac{f'(x)}{g'(x)}$相等，即：

$$\boldsymbol{\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}}$$

2. Solution steps based on L'Hôpital's rule

1. Differentiation of the numerator function: (sinx)′= cosx
2. Differentiating the denominator function: (x)′=1
3. Substitute the limiting conditions into the calculation:

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\cos x}{1}=\cos 0=1$$

3. Theoretical verification of the regression Taylor-McLaughlin expansion

(I) Derivation of the Maclaurin expansion of the sine function

1. Periodicity of derivatives: The higher-order derivatives of a sine function exhibit a cyclic property with a period of 4.

   $(\sin x)' = \cos x$, $(\cos x)' = -\sin x$, $(-\sin x)' = -\cos x$, $(-\cos x)' = \sin x$;
2. Expansion filtering rule: The Maclaurin expansion is the Taylor expansion when a=0 , sin0=0 , and its even-order derivatives take the value of 0 at x=0 , so only the odd-order terms are retained;
3. Standard expansion:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

(ii) Simplifying operations by substituting into limit expressions

Substituting the Maclaurin expansion of sinx into the fraction sinx /x , we perform algebraic simplification:

$$\frac{\sin x}{x} = \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots$$

(III) Consistency between the limit value and the conclusion

As $x \to 0$, the higher-order infinitesimal terms such as x^2 and x^4 in the expansion all approach 0, therefore:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$